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# Positivity and the Axial-Vector Current in Quantum Electrodynamics*

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ABSTRACT

In finite theories of quantum electrodynamics, positivity implies $d > 3$ for the dimension $d$ of an axial-vector current $J_{5\mu}$ with non-zero anomaly. This result is not contradicted in the Johnson-Baker-Willey and Adler models: arguments for the neglect of internal fermion creation and annihilation fail for $J_{5\mu}$- amplitudes because illegal skeleton expansions are involved.

Attempts[1,2] to include an axial-vector current $J_{5\mu}$ in finite theories[3-5] of quantum electrodynamics (QED) have produced unexpected difficulties. In particular, the annihilation condition[6]

$$J_\mu(0)|0\rangle = 0 \ , \quad (\text{fermion mass } m = 0), \tag{1}$$

for the electromagnetic current $J_\mu$ seems to imply the result[1,2]

$$\langle 0|J_\alpha J_\beta J_{5\gamma}|0\rangle = 0 \ , \qquad (m = 0). \tag{2}$$

However, in any theory summed over gauge-invariant subsets of diagrams (with $m \neq 0$), the corresponding anomalous constant $S$ is not renormalized[7]:

$$\partial^\gamma J_{5\gamma} = J_5 + (\alpha S/4\pi)\,[F.\tilde{F}] \quad ,$$

$$S = -\frac{\pi^2}{12}\,\epsilon^{\mu\rho\alpha\beta}\iint d^4x\,d^4y\,x_\mu y_\rho\ T\langle 0|J_\alpha(x)J_\beta(0)J_5(y)|0\rangle = 1 \ , \ (m \neq 0); \tag{3}$$

(the symbol $J_5$ represents a soft pseudoscalar operator, and $[F.\tilde{F}]$ denotes the renormalized gauge-invariant normal-product operator constructed from the electromagnetic field-strength tensor $F_{\alpha\beta}$ and its dual $\tilde{F}_{\alpha\beta}$). According to Wilson's analysis[8] of the anomaly, Eq. (2) implies[9] $S = 0$, a result which is not compatible with (3).

We have already given an extensive discussion of this problem and related difficulties elsewhere.[10] This abbreviated version, unencumbered with side issues, serves to emphasize the main conclusions:

(a) Positivity and Eq. (3) imply that $d$, the dimension of $J_{5\mu}$, is <u>greater</u> than 3; in that case, Eqs. (1) and (3) are compatible, and Eq. (2) is incorrect.

(b) In the Johnson-Baker-Willey[4] (JBW) and Adler[5] models, the argument that internal fermion creation and annihilation may be asymptotically neglected cannot be applied to $J_{5\mu}$-amplitudes because it involves the use of an illegal skeleton expansion.

Instead of setting m equal to zero, we consider products of smeared gauge-invariant operators such as

$$J_\mu = J_\mu(\rho\,;f) = \int d^4x\, f(x)\, J_\mu(\rho x) \quad ,$$

$$\bar{J}_\mu = J_\mu(\rho\,;f^*) \quad , \qquad (\rho > 0\,,\ \ f = \text{any test function}), \tag{4}$$

in the short-distance limit $\rho \to 0$ of the massive theory. Vectors $|\Psi\rangle, |\phi\rangle, \ldots$ generated by applying these operators to the vacuum state vector $|0\rangle$ obey the Schwarz inequality

$$|\langle\Psi|\phi\rangle|^2 \quad \leqslant \quad \langle\Psi|\Psi\rangle\langle\phi|\phi\rangle \qquad ,$$

of which the result

$$\langle 0|\bar{J}_{5\gamma} J_{5\gamma}|0\rangle \quad \geqslant \quad |\langle 0|J_\alpha J_\beta J_{5\gamma}|0\rangle|^2 / \langle 0|J_\alpha J_\beta \bar{J}_\beta \bar{J}_\alpha|0\rangle \tag{5}$$

is a special case. Since the finiteness condition

$$\lim_{\rho\to 0} \ \rho^6 \langle 0|J_\alpha J_\beta|0\rangle \ = \ 0$$

and positivity imply[11]

$$\lim_{\rho\to 0} \ \rho^{12} \langle 0|J_\alpha J_\beta \bar{J}_\beta \bar{J}_\alpha|0\rangle \ = \ 0 \quad , \tag{6}$$

and Wilson's condition[8]

$$S \neq 0 \quad \Rightarrow \quad \lim_{\rho\to 0} \ \rho^9 \langle 0|J_\alpha J_\beta J_{5\gamma}|0\rangle \ \neq \ 0 \tag{7}$$

remains valid in QED , Eq. (5) leads directly to the conclusion[12]

$$\rho^6 \langle 0|\bar{J}_{5\gamma} J_{5\gamma}|0\rangle \ \to \ \infty \quad , \quad (\rho \to 0\,). \tag{8}$$

In other words, the dimension of $J_{5\mu}$ is greater than 3.

Now we can see why Eq. (2) is not correct. The annihilation condition

$$\langle\phi\,|J_\beta J_\alpha|0\rangle \ = \ 0 \quad , \quad (m \ = \ 0) \tag{9}$$

must be restricted to states $|\phi\rangle$ which possess a strongly convergent zero-mass limit:

$$\lim_{m\to 0} \ \langle\phi\,|\,\phi\rangle \quad < \quad \infty \qquad . \tag{10}$$

In particular, the choice

$$|\phi\rangle \ = \ J_{5\gamma}|0\rangle / \{\langle 0|\bar{J}_{5\gamma} J_{5\gamma}|0\rangle\}^{1/2}$$

is legitimate, but Eqs. (8) and (10) do not permit substitution of $J_{5\gamma}|0\rangle$ for $|\phi\rangle$ in Eq. (9). What this example shows is that construction of the $\rho \to 0$ limit with $m \neq 0$ gives a precise meaning to the term "zero-mass QED".

We now restrict our attention to the models[4,5] of finite QED considered by Adler et al.[1] In addition to the usual renormalization-group (or Callan-Symanzik[13]) equations, the following argument of Baker and Johnson[6] is assumed to be valid: The leading asymptotic behavior of an amplitude is not influenced by classes of subgraphs which sum to

$$\tilde{\pi}_{2n} = T\langle 0|J_{\mu_1} \ldots J_{\mu_{2n}}|0\rangle \tag{11}$$

for $n \geq 2$, because positivity implies the constraint

$$\tilde{\pi}_{2n} \longrightarrow 0 \ , \quad (n \geq 2) \tag{12}$$

in the limit $m \to 0$. Using this argument, Baker and Johnson[6] and Adler[5] have shown that, at the eigenvalue, single-fermion-loop contributions to $\langle 0|J_\alpha J_\beta|0\rangle$ and $\tilde{\pi}_{2n}$ also satisfy the finiteness condition and Eq. (12) respectively.

However, a difficulty seems to arise for the proper amplitude $\tilde{R}_{\alpha\beta\gamma}$ for $J_{5\gamma}$ to couple to two photons. The dimension of $J_{5\gamma}$ is given by

$$d = 3 + \chi(\alpha_e) \ , \tag{13}$$

where the function

$$\chi(\alpha) = 3\alpha^2/2\pi^2 + O(\alpha^3)$$

appears in asymptotic Callan-Symanzik equations such as[14]

$$\left[m\frac{\partial}{\partial m} + \alpha\beta(\alpha)\frac{\partial}{\partial\alpha} + \chi(\alpha)\right]\tilde{R}_{\alpha\beta\gamma} \sim 0 \ , \tag{14}$$

and $\alpha_e$ is the first zero of the Callan-Symanzik function $\beta(\alpha)$.[15] By examining the cutoff-dependence of graphs contributing to the unrenormalized amplitude $R_{\alpha\beta\gamma}$, one can readily see that the set of iterated $\gamma$-$\gamma$ scattering subdiagrams coupling to a bare triangle (Fig. 1) is responsible for the presence of the function $\chi(\alpha)$ in Eq. (14). If one argues that the $\gamma$-$\gamma$ scattering subgraphs are asymptotically negligible at the eigenvalue, only the bare triangle graphs would survive[16], and $\chi(\alpha_e)$ would have to vanish, in contradiction with our previous analysis.

This line of reasoning breaks down because the decomposition of $\tilde{R}_{\alpha\beta\gamma}$ into $\gamma$-$\gamma$ scattering subgraphs constitutes an <u>illegal skeleton expansion.</u> In order to apply the $m \to 0$ limit, it is necessary that $\tilde{R}_{\alpha\beta\gamma}$ (Fig. 2a) be written in the form $\int d^4k\, I_{\alpha\beta\gamma}(k)$, where the integrand

$$I_{\alpha\beta\gamma} = (2\pi^4)^{-1} (\tilde{\pi}_4 \tilde{D}'_F \tilde{D}'_F \tilde{T})_{\alpha\beta\gamma}$$

involves <u>renormalized</u> amplitudes, and that the limit and integral be interchanged. (Here, $\tilde{D}'_F$ is the complete renormalized photon propagator, $\tilde{T}$ is the renormalized amplitude represented by the bare triangle graphs, and $k$ is the photon-loop momentum.) In general, whenever a diagram can be decomposed as in Fig. 2b, the corresponding loop integral <u>diverges</u> logarithmically. When this divergence is removed by the usual methods, the result <u>cannot</u> be written in terms of an integrand proportional to $\tilde{\pi}_4$. Thus, even in the JBW and Adler models of QED, non-canonical scaling of $J_{5\gamma}$-amplitudes is permitted in the asymptotic region because some of the diagrams in Fig. 1 are <u>not</u> asymptotically negligible.[17] Note that the correct skeleton expansion (Fig. 2c) leads to the formula

$$R_{\alpha\beta\gamma} = (2\pi^4)^{-1}\, \mathrm{Tr} \int d^4q\, \tilde{\Gamma}_{5\gamma} \tilde{S}'_F \tilde{S}'_F \tilde{C}_{\alpha\beta} \tag{15}$$

which contains the following renormalized amplitudes: the full electron propagator $\tilde{S}'_F$, proper axial-vector vertex $\tilde{\Gamma}_{5\gamma}$, and proper Compton scattering amplitude[18] $\tilde{C}_{\alpha\beta}$.

It is important to emphasize that we are <u>not</u> challenging applications[4,5] of the Baker-Johnson argument to purely electrodynamic amplitudes, (e.g., $\tilde{\pi}_{2n}$). The legitimacy of skeleton expansions used in these applications can be readily demonstrated with the help of gauge invariance. Troubles with divergent subintegrations do not arise unless new vertices such as $\gamma_\mu\gamma_5$ are introduced.

Having resolved these logical difficulties associated with positivity and the existence of $J_{5\gamma}$ in finite theories of QED, we should conclude by listing potential problems for the theories of JBW[4] and Adler[5]:

(i) Irrespective of which operators exist in the complete theory, it is known[1] that the limit $z \ll x, y$ and the infinite sum over one-fermion-loop contributions to $\langle 0|J_\alpha(x)J_\beta(y)J_\gamma(z)J_\mu(0)|0\rangle$ fail to commute. This is disturbing because it is a basic assumption of renormalization-group theory that this lack of uniformity does not occur for the zero-mass limit.

(ii) There is no guarantee that the summation procedures of JBW and Adler preserve positivity. In particular, a gluon-model vector space generated by currents $\mathcal{F}_\mu$, $\mathcal{F}_{5\mu}$ which describe a non-Abelian symmetry of the fermions <u>cannot</u> possess a positive metric, because $\mathcal{F}_\mu$ and $\mathcal{F}_{5\mu}$ have dimension 3 and obey the constraint[19,20]

$$\rho^6\langle 0|\mathcal{F}_\mu \mathcal{F}_\nu|0\rangle = \rho^6\langle 0|\mathcal{F}_{5\mu}\mathcal{F}_{5\nu}|0\rangle \longrightarrow 0 \ , \ (\rho \to 0). \qquad (16)$$

A more detailed analysis of these matters is given elsewhere.[10]

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8. K. G. Wilson, Phys. Rev. 179, 1499 (1969); R. J. Crewther, Phys.

Rev. Letters 28, 1421 (1972).

9. On this point, we disagree with R. Köberle and N. K. Nielsen, Phys. Rev. D 8, 660 (1973). If the validity of Eq. (2) is assumed, the absorptive parts of $T\langle 0|J_\alpha J_\beta [F.\tilde{F}]|0\rangle$ cannot affect Wilson's analysis because Eq. (3) implies that these terms are insufficiently singular at short distances.

10. R. J. Crewther, S.-S. Shei, and T.-M. Yan, Phys. Rev. D (to be published).

11. This result is usually written as a special case of Eq. (1): $\langle J_\alpha J_\beta J_\gamma J_\delta\rangle = 0$, $(m = 0)$. Instead, it can be obtained directly from the inequality $|\langle JJJJ\rangle|^2 \leq \langle J\bar{J}\rangle \langle \bar{J}\bar{J}\bar{J}JJJ\rangle$, $(m \neq 0)$: since $J_\mu$ has canonical dimension, $\rho^{18}\langle \bar{J}\bar{J}\bar{J}JJJ\rangle$ cannot be singular at $\rho = 0$.

12. Eq. (3) is crucial in the derivation of (8). Positivity alone merely implies $\rho^4\langle 0|J_{5\alpha}J_{5\beta}|0\rangle \not\to 0$; see R. J. Crewther et al., Ref. 10.

13. C. G. Callan, Jr., Phys. Rev. D 2, 1541 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970).

14. See R. Köberle and N. K. Nielsen, Ref. 9.

15. In Ref. 10, we have analyzed asymptotic solutions of the Callan-Symanzik equations when $\alpha_e$ is not a simple zero of $\beta(\alpha)$, (with $\alpha<\alpha_e$). In general, violations of asymptotic scale invariance should be expected for amplitudes such as $\tilde{R}_{\alpha\beta\gamma}$. The rule is: if the naive scale-invariant result is $\rho^\gamma$ in the limit $\rho \to 0$, the correct result is $\rho^{\gamma + \lambda(\rho)}$, with $\lambda(0) = 0$. This refinement can be incorporated in the present discussion without difficulty.

16. S. L. Adler <u>et</u> <u>al</u>. (Ref. 1) observed that the other one-fermion-loop contributions to $\widetilde{R}_{\alpha\beta\gamma}$ are asymptotically negligible.

17. See Ref. 10 for an explicit listing of these diagrams.

18. We omit intermediate states which contain just a single fermion-antifermion pair.

19. This condition seems to be peculiar to the JBW and Adler models.

20. Clearly, we do not want Eq. (16) to be satisfied by the usual $SU(3)\times SU(3)$ currents for hadrons; in other words, the total cross section for $e^+e^- \rightarrow$ hadrons must not decrease faster than $E^{-2}$ as the center-of-mass energy E becomes large. This conclusion is also an obvious consequence of the short-distance analysis of R. J. Crewther, (Ref. 8).

## FIGURE CAPTIONS

Fig. 1: Set of graphs which generates the function $\chi(\alpha)$ in Eq. (14). The two-photon intermediate state is omitted in each set of subgraphs $\pi_4'$. It is understood that graphs with crossed lines are included.

Fig. 2: Decompositions of $\widetilde{R}_{\alpha\beta\gamma}$. (a) Illegal skeleton expansion. (b) Contributions to $\widetilde{R}_{\alpha\beta\gamma}$ for which $\int d^4k\, I_{\alpha\beta\gamma}(k)$ diverges. The divergence appears when all internal momenta above the dotted line become simultaneously large. (c) Correct skeleton expansion; see Eq. (15).

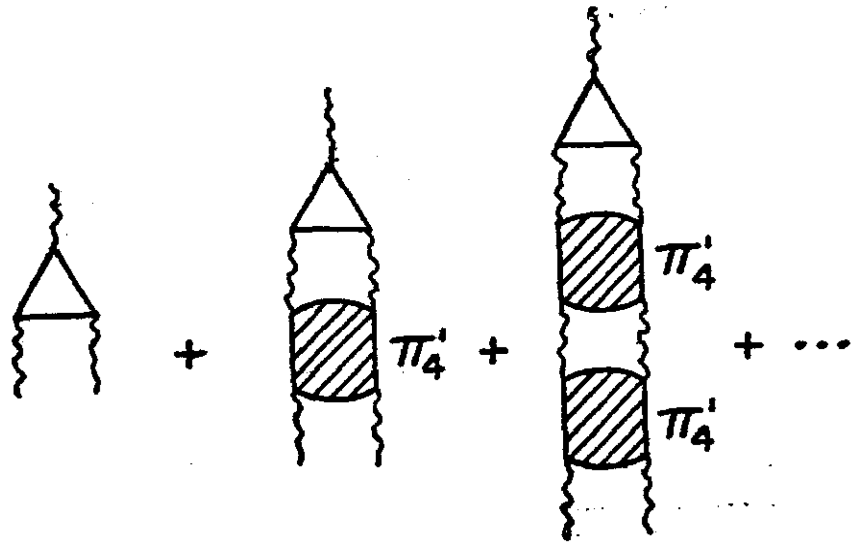


FIG. 1

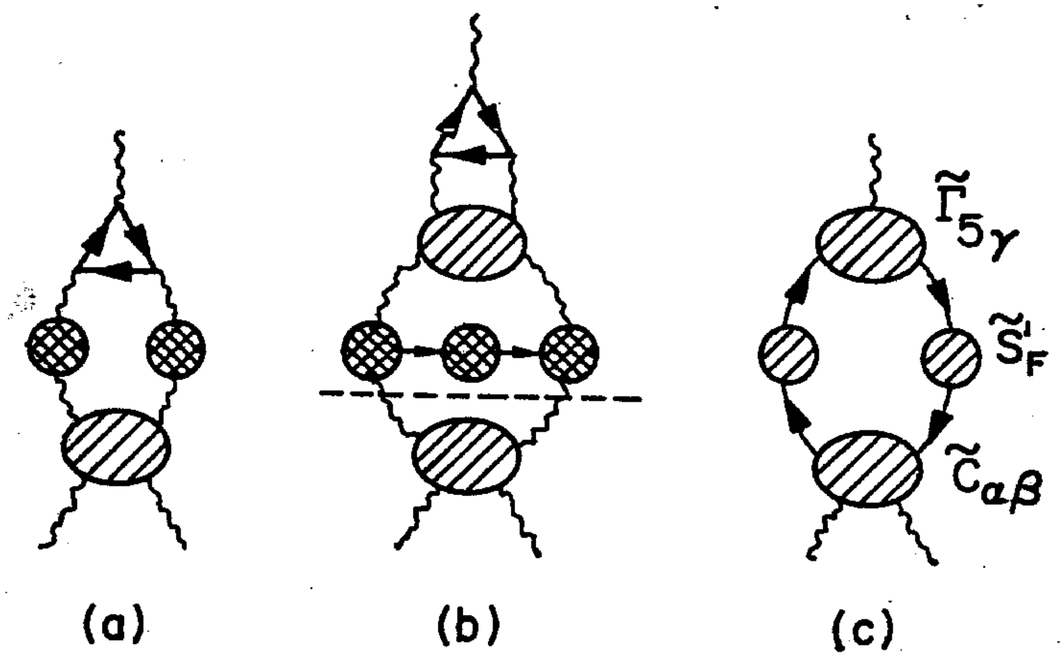


FIG. 2

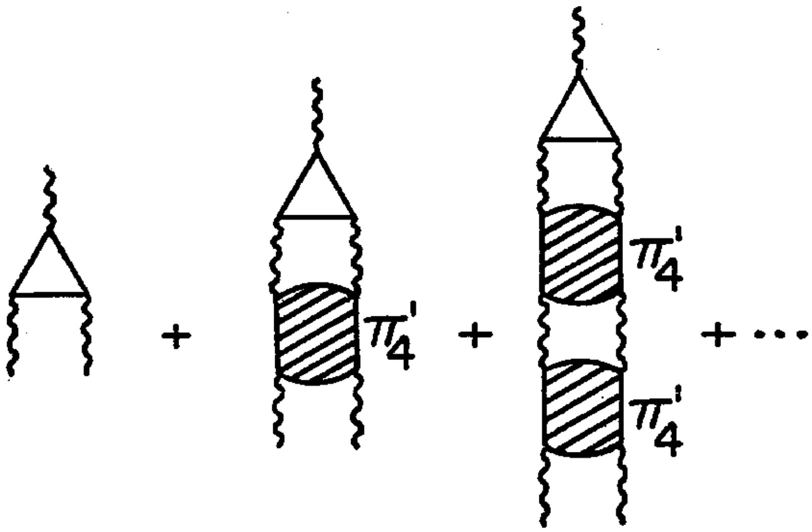


Figure 1

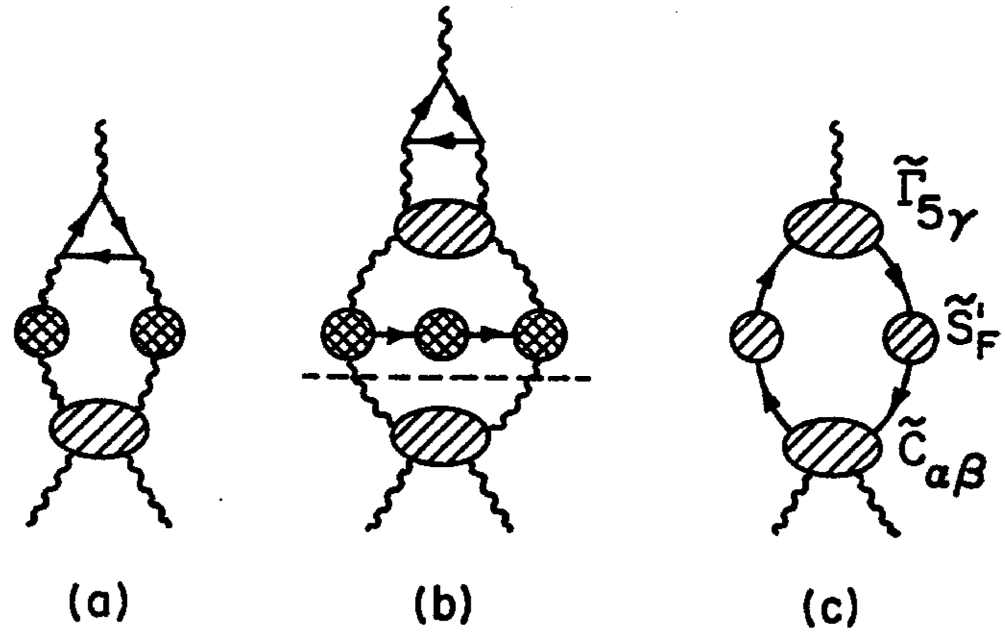


Figure 2